Geometrical description of spin-2 fields

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We show that the torsion of a Cartan geometry can be associated to two spin-2 fields. This structure allows a new approach to deal with the proposal of geometrization of spin-2 fields besides the traditional one dealt with in General Relativity. We use the associated Hilbert-Einstein Lagrangian R for generating a dynamics for the fields.

I. INTRODUCTION

A. Introductory remarks I

From the particle physicists point of view the use of a spin-2 field represented by a second order symmetrical tensor $\varphi_{\mu\nu}$ in order to define a metric tensor $g_{\mu\nu}$ of a Riemannian spacetime seems quite natural. Both quantities have the same tensorial character and the same explicit symmetry. In other words, if one faces the question of how to associate a symmetric second order tensor $\varphi_{\mu\nu}$ that represents a spin-2 field, into a geometrical framework, the lesson we learned from last century leads almost univocally to a prompt answer: one should add this field to the Minkowski metric tensor of the flat background spacetime and generate a curved Riemannian geometry by defining an associated metric tensor by the expression:

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \varphi_{\mu\nu}. \tag{1}$$

Note that this is not an approximation but an exact relation. To obtain from the above expression the corresponding inverse $g^{\mu\nu}$ defined by

$$g^{\mu\nu}\, g_{\nu\alpha} = \delta^{\mu}_{\alpha}$$

one is led to the infinite series

$$g^{\mu\nu} = \eta^{\mu\nu} - \varphi^{\mu\nu} + \varphi_{\mu\alpha}\, \varphi^{\alpha\nu} + \cdots \tag{2}$$

With these definitions it follows the inevitability of a nonlinear process to implement the theory based on the identification equation (1). This kind of geometrical formulation is the basis of Einstein General Relativity (GR). Although Einstein path starts from an a priori geometrical description, there is an equivalent description that is based on a field theoretical formulation. The best way to understand this is to follow the work of Gupta, Deser, Feynmann and others. See [1] for details and further references.

A dynamics within this geometrical scheme is obtained from Hilbert-Einstein action in terms of the scalar of curvature of the associated Riemannian geometry:

$$S = \int \sqrt{-g}\, R\, d^4x. \tag{3}$$

The success of such procedure led to the belief that in 4-dimension there is no way to introduce any other spin-2 field in the geometry. Or, at least, not in such simple and natural way, as is done in the above definition (1).

The purpose of this paper is precisely to discuss on this and to point out the existence of another "natural" way to associate spin-2 fields to the geometry. In other words we will show an alternative approach to geometrize a spin-2 field. Let me stress from the beginning that we will not deal here with an alternative way to describe gravitational effects. Our work here is just to examine the geometrical consequences of the existence of others spin-2 fields besides the gravitational field which is associated to the metric tensor. The motivation of this paper is precisely this: to report on a new form to integrate spin-2 field in the geometry. We will show that the scenario which allows for such task already exists and is realized by means of a well known generalization of the riemannian structure of spacetime just by assuming the presence of torsion in the geometry. In other words, one has to consider Cartan geometry [2], which, besides the metric, contains an antisymmetric torsion $\tau^{\alpha}_{\mu\nu}$.

In order to understand such identification of the spin-2 field with torsion we have to start by recovering a property of field theory established in the first part of the last century by Fierz. The main result states that a spin-2 field can be described in two equivalent ways, which we will call the Einstein-frame and the Fierz-frame representations. The most common one, the Einstein-frame, uses a symmetric second order tensor $\varphi_{\mu\nu}$ to represent the field. In the Fierz-frame this role is played by a third order tensor $F_{\alpha\mu\nu}$, which is antisymmetric in the first pair of indices [3]: $F_{\mu\nu\alpha} = -F_{\nu\mu\alpha}$. The existence of this Fierz-representation opens a new way to associate spin-2 field to the geometry which seems as natural as the one pointed out above. We shall see that this new form of geometrization depends crucially on the uses of Fierz representation of the spin-2 field in terms of a third

rank tensor. The oblivion of Fierz representation along the last fifty years and the success of General Relativity were responsible for the general belief that a symmetric second order tensor is the unique way to represent a spin-2 field and consequently is the reason for which such scheme of geometrization was not noticed before.

Our analysis rest on the simplest generalization of Riemann geometry that deals with an affine connection which is not symmetric. In other words, the connection is written as the sum of the Christoffel symbol and an extra tensor defined by the torsion tensor $\tau^{\alpha}_{\mu\nu}$. This tensor is antisymmetric: $\tau^{\alpha}_{\mu\nu} = -\tau^{\alpha}_{\nu\mu}$. We shall see that as in the case of spin-2 description in terms of a symmetric second order rank, in the case in which the space is endowed with a torsion it will be equally "natural" to associate the torsion to spin-2 tensors in the Fierz representation.

B. Introductory Remarks II

Recently [4] we have analysed the formulation of spin-2 field (massive and massless) in terms of a third order tensor $F_{\alpha\mu\nu}$,. As a consequence of such exam we concluded that in flat Minkowski spacetime both variables are equivalent: the dynamics is the same and the corresponding structure of the consistency of the dynamical equations is completely equivalent. Nevertheless, in the case of a curved spacetime this is no longer true. We have shown that the use of the Fierz-frame seems more compelling, once it yields, through the standard minimal coupling principle, a unique, non-ambiguous description. The use of the Einstein-frame in the passage from the flat Minkowskii background to the curved Riemannian one in General Relativity, introduces ambiguities which come from the non commutativity of the covariant derivatives. In [4] we have shown that the use of the Fierz-frame gives an unambiguous minimal coupling treatment for equations that are equivalent to those studied by Aragone-Deser and Buchbinder et al. Thus, not only the Fierz frame is equivalent to Einstein frame to describe spin-2 fields but, more than this, it avoids the arbitrariness and inconsistency that exists in the standard formulation of a spin-2 field coupled to gravity in Einstein-frame. Besides, the superiority of the Fierz frame appears more explicitly in the combined set of equations for spin-2 field and gravity: it preserves the standard Einstein equations of motion, whilst maintaining the correct degrees of freedom for the spin-2 field.

The idea that torsion can be associated to fundamental fields has been examined previously (see for instance [5]) in a tentative to incorporate torsion as a manifestation of a completely antisymmetric field typical of string theories. In the present paper we intend to present another way to look into this problem by incorporating two spin-2 tensors into the Cartan geometrical scheme.

In order to exhibit more clearly the properties of this identification we start with the simplest geometrical structure embodied with torsion and a Minkowski metric tensor $\eta_{\mu\nu}$. The dynamics mimics General Relativity using the scalar of curvature as a natural generalization of Hilbert-Einstein action. In this particular case the curvature is a consequence only of the presence of the torsion part of the affine connection, since the Riemannian part of the curvature –which depends on the metric tensor and on the associated Christoffel symbol – vanishes.

C. Synopsis

In this paper we will deal with Cartan geometry. In section II we present a set of definitions and equations which will be needed in this paper. Section III deals with the particular case of restricted Cartan geometry in which the degrees of freedom of the torsion field are reduced from 24 to 10. We present the affine connection and the associated curvature tensor. We use the scalar of curvature of the restricted case in order to produce a dynamics for torsion. We show that this dynamics coincides with the standard dynamics for a spin-2 field, that is, the linearized Einstein equation of GR. In the next section we generalize this formalism and deal with 20 degrees of freedom. We use the same structure of the scalar of curvature to produce a dynamics for torsion and we show that it is nothing but the same dynamics for two non-interacting spin-2 fields. In the last section we deal with the interaction of these spin-2 fields with matter using the minimal coupling principle. We end with some comments on this and some perspectives for future work.

II. SOME MATHEMATICAL MACHINERY

A. Fierz representation of spin-2 field

We define a three-index tensor $F_{\alpha\beta\mu}$ which is anti-symmetric in the first pair of indices and obeys the cyclic identity, that is

$$F_{\alpha\mu\nu} + F_{\mu\alpha\nu} = 0 \tag{4}$$

$$F_{\alpha\mu\nu} + F_{\mu\nu\alpha} + F_{\nu\alpha\mu} = 0. \tag{5}$$

This last expression means that the dual of $F_{\alpha\mu\nu}$ is trace-free:

$$\overset{*}{F}{}^{\alpha\mu}{}_{\mu} = 0, \tag{6}$$

where the asterisk represents the dual operator, defined in terms of the completelly anti-symmetric object $\eta_{\alpha\beta\mu\nu}$ by

$$\overset{*}{F}{}^{\alpha\mu}{}_{\lambda} \equiv \frac{1}{2}\,\eta^{\alpha\mu}{}_{\nu\sigma}\,F^{\nu\sigma}{}_{\lambda}.$$

The dual of the Levi-Civita object allows the introduction of the tensor $g_{\alpha\beta\mu\nu}$ once we have the equality

$$g_{\alpha\beta\mu\nu} = -\,\eta^{*}_{\alpha\beta\mu\nu}, \tag{7}$$

where

$$g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu}\, g_{\beta\nu} - g_{\alpha\nu}\, g_{\beta\mu}. \tag{8}$$

B. The case of a single spin-2 field

The field $F_{\alpha\mu\nu}$ has 20 independent components. In order to eliminate the extra 10 independent components and allow it to represent a single spin-2 field, we impose an additional requirement contained in the following lemma[8]: The necessary and sufficient condition for $F_{\alpha\mu\nu}$ to represent an unique spin-2 field is

$$\overset{*}{F}{}^{\alpha(\mu\nu)}{}_{,\alpha} = 0. \tag{9}$$

We represent the symmetrization symbol by $A_{(\mu\nu)} \equiv A_{\mu\nu} + A_{\nu\mu}$. We use an analogous form for the anti-symmetrization symbol: $[x, y] \equiv xy - yx$.

We will call a tensor that satisfies conditions (4), (5) and (9) a **Fierz tensor**.

Condition (9) implies that there exists a symmetric second order tensor $A_{\mu\nu} = A_{\nu\mu}$ such that we can write

$$2F_{\alpha\mu\nu} = A_{\nu\alpha,\mu} - A_{\nu\mu,\alpha} - g_{\alpha\mu\nu\epsilon}F^{\epsilon}. \tag{10}$$

The factor 2 in the l.h.s. is introduced for convenience.

$$F_{\alpha} \equiv F_{\alpha\mu\nu}g^{\mu\nu} = A_{,\alpha} - A_{\alpha}{}^{\lambda}{}_{,\lambda}, \tag{11}$$

and $A \equiv A_{\mu\nu}g^{\mu\nu}$.

When a Fierz tensor is written under the form given in equation (10) we will say that it is in the Einstein frame. This formula can be made covariant and generalized trivially for arbitrary system of coordinates.

A Fierz tensor $F_{\alpha\mu\nu}$ satisfies the identity

$$F^{\alpha}{}_{(\mu\nu),\alpha} \equiv -\,G^{L}{}_{\mu\nu}, \tag{12}$$

where $G^{L}{}_{\mu\nu}$ is the linearized Einstein operator defined in terms of the symmetric tensor $A_{\mu\nu}$ by

$$G^{L}{}_{\mu\nu} \equiv \Box\, A_{\mu\nu} - A^{\epsilon}{}_{(\mu,\nu)\,,\epsilon} + A_{,\mu\nu} - g_{\mu\nu}\,\left(\Box A - A^{\alpha\beta}{}_{,\alpha\beta}\right). \tag{13}$$

C. Dynamics

We limit all our considerations in the present paper to a dynamics for the Fierz field which is linear. The most general linear theory comes from a combination of the invariants one can construct with the field. There are two[9] of them which we represent by X and Y :

$$\begin{aligned} X &\equiv F_{\alpha\mu\nu}\, F^{\alpha\mu\nu} \\ Y &\equiv F_{\mu}\, F^{\mu}. \end{aligned} \tag{14}$$

The standard equation for the massless spin-2 field, the linearization of Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is given by

$$G^{L}{}_{\mu\nu} = 0, \tag{15}$$

or, in an equivalent way, using the identity (12)

$$F^{\lambda(\mu\nu)}{}_{,\lambda} = 0. \tag{16}$$

The corresponding action takes the form

$$S = \frac{1}{k}\int (X - Y)\, \mathrm{d}^4x. \tag{17}$$

Note that the Fierz tensor has dimensionality (lenght)$^{-1}$. Thus the constant k has dimensionality (energy)$^{-1}$ (lenght)$^{-1}$.

D. The case of two spin-2 fields

In the general case, in which condition (9) is not satisfied, $F^{\lambda\mu\nu}$ represents two spin-2 fields. In this case $F_{\epsilon\nu\mu}$ has 20 independent components. We use the Fierz decomposition in terms of two symmetric second order tensors $A_{\mu\nu}$ and $B_{\mu\nu}$ that is:

$$F_{\alpha\beta\mu} = A_{\alpha\beta\mu} + \frac{1}{2}\,\eta_{\alpha\beta}{}^{\rho\sigma}\, B_{\rho\sigma\mu}, \tag{18}$$

where

$$2\, A_{\alpha\beta\mu} \equiv A_{\mu[\alpha,\beta]} - g_{\alpha\beta\mu\epsilon}\, A^{\epsilon} \tag{19}$$

$$2\, B_{\alpha\beta\mu} \equiv B_{\mu[\alpha,\beta]} - g_{\alpha\beta\mu\epsilon}\, B^{\epsilon} \tag{20}$$

$$A_{\lambda} \equiv A_{,\lambda} - A_{\lambda}{}^{\mu}{}_{,\mu} \tag{21}$$

$$B_{\lambda} \equiv B_{,\lambda} - B_{\lambda}{}^{\mu}{}_{,\mu}, \tag{22}$$

with $A \equiv A_{\mu\nu}\, g^{\mu\nu}$ and $B \equiv B_{\mu\nu}\, g^{\mu\nu}$. Note that the trace and the pseudo-trace of the tensor $F_{\alpha\beta\mu}$ are, respectively,

$$F_{\alpha\beta}{}^{\beta} = A_{\alpha} \tag{23}$$

$$F^{*}_{\alpha\beta}{}^{\beta} = -\,B_{\alpha}. \tag{24}$$

E. Torsion

Let us now consider a four dimensional spacetime endowed with a metric tensor $g_{\mu\nu}$ and a non symmetric affine connection $\Gamma^{\alpha}_{\mu\nu}$ that defines a covariant derivative. This structure is usually called a Cartan space. For an arbitrary vector V^{μ}, the covariant derivative is defined by:

$$V^{\mu}{}_{;\nu} \equiv V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\nu\alpha} V^{\alpha}. \tag{25}$$

The torsion tensor $\tau^{\alpha}_{\mu\nu}$ is given by

$$\tau^{\alpha}_{\mu\nu} \equiv \frac{1}{2}\left(\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}\right). \tag{26}$$

It is worthwhile to decompose torsion tensor into its irreducible components by setting:

$$\tau^{\alpha}_{\mu\nu} = L^{\alpha}_{\mu\nu} + \frac{1}{3}\left(\delta^{\alpha}_{\mu}\,\tau_{\nu} - \delta^{\alpha}_{\nu}\,\tau_{\mu}\right) - \frac{1}{3}\,\eta_{\mu\nu}{}^{\alpha\lambda}\,\tau^{*}_{\lambda} \tag{27}$$

The quantity $\tau_{\mu} = \tau^{\alpha}{}_{\alpha\mu}$ is the trace while $\tau^{*}_{\mu} = \tau^{\alpha *}{}_{\alpha\mu}$ is the pseudo-trace. Note that the condition of metricity is satisfied, that is

$$g_{\mu\nu;\alpha} = 0. \tag{28}$$

III. RESTRICTED CARTAN GEOMETRY (RCG)

In this section we will deal with the simple case of a pseudo-traceless Cartan geometry, that is we set:

$$\tau^{*}_{\alpha} = 0. \tag{29}$$

This condition diminishes the 24 degrees of freedom of torsion to 20.

It is useful to represent torsion by an associated quantity $F_{\mu\nu}{}^{\alpha}$ defined by the combination

$$\tau^{\alpha}{}_{\mu\nu} - g^{\alpha\epsilon}{}_{\mu\nu}\,\tau_{\epsilon} = F_{\mu\nu}{}^{\alpha}. \tag{30}$$

It follows that this tensor $F_{\mu\nu\alpha}$ is anti-symmetric in the first two indices:

$$F_{\mu\nu\alpha} = -F_{\nu\mu\alpha}. \tag{31}$$

Besides, since the torsion tensor has no pseudo-trace it follows, from definition (30) the additional symmetry

$$F_{\mu\nu\alpha} + F_{\nu\alpha\mu} + F_{\alpha\mu\nu} = 0. \tag{32}$$

The tensor $F_{\mu\nu\alpha}$ so defined has, like the torsion with vanishing pseudo trace, 20 degrees of freedom.

Thus, it seems natural to identify this tensor with two spin-2 fields in the Fierz representation. In this way we obtain a natural framework to connect spin-2 fields with the geometry. This is our goal in the present paper[10].

Let us simplify here our analysis and consider the case in which one of these tensors, say $B_{\mu\nu}$, vanishes. We postpone the exam of the full case for the next section. We set

$$2\,F_{\alpha\beta\mu} = A_{\mu[\alpha,\beta]} + F_{[\alpha}\eta_{\beta]\mu}, \tag{33}$$

In order to analyse the curvature properties of this RCG let us remind that the important quantity is not torsion itself but the so called contortion, $K_{\alpha\beta\mu}$, defined by

$$\Gamma^{\epsilon}_{\mu\nu} = \left\{{}^{\epsilon}_{\mu\nu}\right\} + K^{\epsilon}{}_{\mu\nu} \tag{34}$$

In this section we restrict our analysis to the case in which the riemannian geometry is flat. In other words, we can choose conveniently the coordinate system in such a way that the Christoffel symbol vanishes. This choice makes no restriction in all calculations and all procedure throughout the whole paper is completely covariant.

Using equation (30) the contortion is written as

$$K^{\epsilon}{}_{\mu\nu} = 2F^{\epsilon}{}_{\nu\mu} + F_{\nu}\,\delta^{\epsilon}_{\mu} - F^{\epsilon}\,\eta_{\mu\nu} \tag{35}$$

or, in terms of the field $A_{\mu\nu}$:

$$K_{\epsilon\mu\nu} = A_{\epsilon\mu,\nu} - A_{\mu\nu,\epsilon}, \tag{36}$$

from which follows the antisymmetry $K_{\epsilon\mu\nu} + K_{\nu\mu\epsilon} = 0$.

A. Curvature tensor in RCG

The curvature in an affine geometry is defined by

$$R^{\alpha}{}_{\sigma\beta\lambda} = \Gamma^{\alpha}_{\beta\sigma,\lambda} - \Gamma^{\alpha}_{\lambda\sigma,\beta} + \Gamma^{\alpha}_{\lambda\rho}\,\Gamma^{\rho}_{\beta\sigma} - \Gamma^{\rho}_{\lambda\sigma}\,\Gamma^{\alpha}_{\beta\rho}. \tag{37}$$

Since the metric tensor is Minkowskian this tensor contains only the contributions that come from the torsion. The contracted curvature tensor is then given by

$$R_{\mu\nu} = K^{\alpha}{}_{\alpha\mu,\nu} - K^{\alpha}{}_{\nu\mu,\alpha} + K^{\alpha}{}_{\nu\rho}\,K^{\rho}{}_{\alpha\mu} - K^{\alpha}{}_{\alpha\rho}\,K^{\rho}{}_{\nu\mu}, \tag{38}$$

Using equation (36) this curvature tensor can be re-written as

$$R_{\mu\nu} = \Box A_{\mu\nu} - A^{\alpha}{}_{(\mu,\nu)\alpha} + A_{,\mu\nu} + [KK]_{\mu\nu} \tag{39}$$

where

$$[KK]_{\mu\nu} \equiv K^{\alpha}{}_{\nu\rho}\,K^{\rho}{}_{\alpha\mu} - K^{\alpha}{}_{\alpha\rho}\,K^{\rho}{}_{\nu\mu}. \tag{40}$$

Then using the decomposition in terms of the field $A_{\mu\nu}$ as above this can be re-written as

$$\begin{aligned}[KK]_{\mu\nu} \;&\equiv\; \left(A_{\nu\alpha,\rho} - A_{\nu\rho,\alpha}\right)\left(A^{\alpha\rho,\mu} - A_{\mu}{}^{\alpha,\rho}\right) \\ &-\; \left(A_{,\alpha} - A_{\alpha}{}^{\epsilon}{}_{,\epsilon}\right)\eta^{\alpha\rho}\,\left(A_{\nu\rho,\mu} - A_{\nu\mu,\rho}\right).\end{aligned} \tag{41}$$

From this expression it follows immediately that the trace $[KK] \equiv [KK]_{\mu\nu}\,\eta^{\mu\nu}$, is

$$[KK] = -2\,U \tag{42}$$

where U is the invariant

$$U \equiv F_{\alpha\beta\mu}\, F^{\alpha\beta\mu} - F_\alpha\, F^\alpha. \tag{43}$$

Then, for the scalar of the curvature in RCG we obtain

$$R = 2\,\Box A - 2\,A^{\alpha\beta}{}_{,\alpha\beta} - 2\,U. \tag{44}$$

or, using the equation (33)

$$R = 2\,\left(F^\alpha{}_{,\alpha} - U\right) \tag{45}$$

B. Dynamics in RCG

It seems natural to examine the dynamical torsion by choosing for the Lagrangian function the scalar of curvature. From what we have shown above this yields,

$$S = \int R\, d^4x = -2\int U\, d^4x \tag{46}$$

up to a total divergence.

This dynamics is precisely the standard one proposed by Fierz [3] to describe a spin-2 field and corresponds to the linear limit of Einstein General Relativity. Indeed, from the above Lagrangian[11] by varying $A_{\mu\nu}$ it follows:

$$\delta S = -\int 2F^{\alpha(\mu\nu)}{}_{,\alpha}\,\delta A_{\mu\nu}\, d^4x \tag{47}$$

Since we have the identity

$$F^{\alpha\mu\nu}{}_{,\alpha} = \frac{1}{2}\,F^{\alpha(\mu\nu)}{}_{,\alpha} = -\,\frac{1}{2}\,G^L{}_{\mu\nu}, \tag{48}$$

the equation of motion can be re-written as

$$\delta S = 2\int G^L{}_{\mu\nu}\,\delta A^{\mu\nu}\, d^4x, \tag{49}$$

where

$$G^L{}_{\mu\nu} \equiv \Box\, A_{\mu\nu} - A^\epsilon{}_{(\mu,\nu)\,,\epsilon} + A_{,\mu\nu} - \eta_{\mu\nu}\,\left(\Box A - A^{\alpha\beta}{}_{,\alpha\beta}\right), \tag{50}$$

where the label L means the linear part of the Einstein equation for GR[12]. This accomplishes the proof of the following two assertions:

- Restricted torsion (10 degrees of freedom) describes a symmetric second order tensor $A_{\mu\nu}$;
- The dynamics generated by the scalar of curvature (Hilbert-Einstein action) yields the standard linear equation for the spin-2 field $A_{\mu\nu}$.

The next step is to go beyond this property to include other degrees of freedom for torsion. We shall prove now that doubling the number of variables from 10 to 20 causes the appearance of a second spin-2 field.

IV. CARTAN GEOMETRY

The generalization of the above formulation starts by the modification of the representation of the torsion as in equation (30) by the following one:

$$\tau^\alpha{}_{\mu\nu} = F_{\mu\nu}{}^\alpha + \frac{1}{2} g^{\alpha\epsilon}{}_{\mu\nu}\, A_\epsilon - \eta_{\mu\nu}{}^{\alpha\epsilon}\, B_\epsilon \tag{51}$$

in which the trace and the pseudo-trace are given, respectively, by

$$\tau_\alpha = \frac{1}{2}\,F_\alpha = \frac{1}{2}\,A_\alpha \tag{52}$$

$$\tau^*_\alpha = 4B_\alpha. \tag{53}$$

The expression (51) restricts the degrees of freedom of the torsion only to 20 independent quantities. We limit all our analysis to this case in order to deal only with two spin-2 fields. For the contortion, expression (35) becomes

$$K_{\epsilon\mu\nu} = 2F_{\epsilon\nu\mu} + g_{\epsilon\nu\mu\lambda}\, A^\lambda + \eta_{\epsilon\nu\mu\lambda}\, B^\lambda \tag{54}$$

or

$$K_{\epsilon\mu\nu} = 2A_{\epsilon\nu\mu} + \eta_{\epsilon\nu}^{\rho\sigma}\, B_{\rho\sigma\mu} + g_{\epsilon\nu\mu\lambda}\, A^\lambda + \eta_{\epsilon\nu\mu\lambda}\, B^\lambda \tag{55}$$

In this case we use the decomposition (18) in terms of two fields and $A_{\epsilon\nu\mu}$ and $B_{\epsilon\nu\mu}$ are defined in (19) and (20).

The scalar of curvature is given by

$$R = K^\alpha{}_{\alpha\mu,\nu} - K^\alpha{}_{\mu\nu,\alpha}\,\eta^{\mu\nu} + [KK]. \tag{56}$$

Let us evaluate $[KK]$. From the above decomposition equation (54) we find:

$$K^\alpha{}_{\alpha\mu} = F_\mu \tag{57}$$

$$K_{\mu\alpha\beta}\,\eta^{\alpha\beta} = -\,F_\mu \tag{58}$$

$$\begin{aligned} K_{\alpha\mu\rho}\, K^{\rho\alpha\mu} &= 2F_{\alpha\rho\mu}\,\left(F^{\rho\mu\alpha} + F^{\mu\alpha\rho}\right) - 8\,A^\alpha\, F_\alpha \\ &+\ 6\,F_\alpha\, F^\alpha + 4\,A_\alpha\, A^\alpha + 2\,B_\alpha\, B^\alpha \end{aligned} \tag{59}$$

Collecting all this yields

$$\begin{aligned} [KK] &= -2B_{\alpha\beta\mu}\, B^{\alpha\beta\mu} + 2B_{\alpha\beta\mu}\, B^{\alpha\beta\mu} \\ &-\ 2\,B_\alpha\, B^\alpha + 2\,A_\alpha\, A^\alpha. \end{aligned} \tag{60}$$

Then, for the scalar of curvature we find

$$R = 2\,A^\alpha{}_{,\alpha} - 2U[A_{\alpha\beta}] + 2U[B_{\alpha\beta}]. \tag{61}$$

where the functional U is given by equation (43), that is

$$U[A_{\alpha\beta}] \equiv A_{\alpha\beta\mu}\, A^{\alpha\beta\mu} - A_\alpha A^\alpha,$$

and

$$U[B_{\alpha\beta}] \equiv B_{\alpha\beta\mu}\, B^{\alpha\beta\mu} - B_\alpha B^\alpha.$$

The important point to stress here concerns the relative signs of $U[A_{\alpha\beta}]$ and $U[B_{\alpha\beta}]$, a property of Cartan geometry in the two spin-2 representation.

Then the dynamics

$$S = \int R d^4x \tag{62}$$

provides, for independent variation of the variables $A_{\mu\nu}$ and $B_{\mu\nu}$, equations of two independent non-interacting spin-2 fields:

$$\begin{aligned} G^L{}_{\mu\nu}(A) &\equiv \Box A_{\mu\nu} - A^\epsilon{}_{(\mu,\nu),\epsilon} + A_{,\mu\nu} \\ &- \eta_{\mu\nu}\left(\Box A - A^{\alpha\beta}{}_{,\alpha\beta}\right) = 0. \end{aligned} \tag{63}$$

$$\begin{aligned} G^L{}_{\mu\nu}(B) &\equiv \Box B_{\mu\nu} - B^\epsilon{}_{(\mu,\nu),\epsilon} + B_{,\mu\nu} \\ &- \eta_{\mu\nu}\left(\Box B - B^{\alpha\beta}{}_{,\alpha\beta}\right) = 0. \end{aligned} \tag{64}$$

It is worth to say that the curvature R of the Cartan geometry, within the framework we are developing here, yields linear equations of motion for the associated spin-2 fields. This property of linearity is typical for models in which the torsion is associated to fundamental fields. The association of spin-2 to the metric, as in GR, leads inevitably to a nonlinear theory.

V. GENERALIZATION TO CURVED RIEMANNIAN BACKGROUND

All the above analysis was carried out in the case in which the metric tensor has no Riemannian curvature. This means that the metric tensor is identified with the Minkowski geometry. The general case in which the Riemannian geometry is not flat is straightforward and causes no difficulty. The attentive reader could be concerned at this point with the known difficulty of coupling spin-2 fields with an arbitrary Riemannian geometry. The uses of the Fierz frame gives the answer to overcome such difficulty as was shown recently [7]. In this case the total connection is given by

$$\Gamma^\alpha_{\mu\nu} = \left\{{}^{\,\alpha}_{\mu\nu}\right\} + K^\alpha_{\mu\nu} \tag{65}$$

The Christoffel symbol $\left\{{}^{\,\alpha}_{\mu\nu}\right\}$ is constructed in terms of the metric tensor $g_{\mu\nu}$ and its derivatives; $K^\alpha_{\mu\nu}$ is the contortion defined previously. The curvature tensor contains an additional term:

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + K^\alpha{}_{\alpha\mu;\nu} - K^\alpha{}_{\nu\mu;\alpha} + K^\alpha{}_{\nu\rho}\, K^\rho{}_{\alpha\mu} - K^\alpha{}_{\alpha\rho}\, K^\rho{}_{\nu\mu}, \tag{66}$$

where $\hat{R}_{\mu\nu}$ is the contracted curvature tensor of the riemannian sector of the curvature. The covariant derivative ; must be taken in this sector. The complete dynamics in such general case is provided by the scalar R:

$$R = \hat{R} + 2\, A^\alpha{}_{;\alpha} - 2U[A_{\alpha\beta}] + 2U[B_{\alpha\beta}]. \tag{67}$$

We postpone the complete analysis of this general case for another paper.

VI. THE INTERACTION WITH MATTER

The fundamental constituents of matter are associated to elementary particles that have non integer spins like quarks and leptons. This property will lead us to concentrate here in the analysis of the coupling of spin-2 fields $A_{\mu\nu}$ and $B_{\mu\nu}$ to spinors $\Psi(x)$.

In order to understand the coupling of the matter field Ψ with the spin-2 displayed in torsion let us review briefly the similar situation of coupling such a field Ψ with gravity. In other words let us analyse the coupling of spin-2 to matter in the Einstein-frame and in the Fierz-frame.

A. The gravitational interaction of the fermionic field

In the standard minimal coupling framework the interaction of matter to the spin-2 fields, within the geometrization scheme that we presented in the previous chapters, consists simply in the operation of analysis of Dirac equation of motion for a spinor embedded in the modified geometry: a Riemannian one in case of gravity (the Einstein frame) and a Cartan geometry in case of the extra two spin-2 fields (the Fierz frame). This is made by the introduction of a covariant derivative of the spinor. In this section we deal only with the Riemann sector. We set

$$\nabla_\mu \Psi \equiv \partial_\mu \Psi - \Gamma_\mu \Psi \tag{68}$$

where the Fock-Ivanenko spinor connection is given in the standard way:

$$\Gamma_\mu = -\frac{1}{4}\left(e^\lambda_b \partial_\mu e^a_\lambda - \left\{{}^{\,a}_{b\,\mu}\right\}\right)\sigma^b{}_a. \tag{69}$$

and

$$\sigma^b_a \equiv \frac{1}{2}\left(\gamma^b\gamma_a - \gamma_a\gamma^b\right) \tag{70}$$

In this expression e^λ_b is a set of orthogonal tetrads and

$$\left\{{}^{\,a}_{b\,\mu}\right\} \equiv \left\{{}^{\,\alpha}_{\beta\,\mu}\right\} e^a_\alpha\, e^\beta_b, \tag{71}$$

The uses of the minimal coupling principle yields for the action of the spinor the form

$$\begin{aligned} S &= \frac{i}{2}\int \sqrt{-g}\,(\bar\Psi\gamma^\alpha\,\nabla_\alpha\,\Psi - \bar\nabla_\alpha\bar\Psi\,\gamma^\alpha\,\Psi)d^4x \\ &- \int \sqrt{-g}\,\mu^2\,\bar\Psi\Psi d^4x \end{aligned} \tag{72}$$

In other words the minimal coupling principle yields for the gravitational interacting action:

$$S_{int} = \int \sqrt{-g}\, T_{\mu\nu} g^{\mu\nu} d^4x \tag{73}$$

where the energy-momentum tensor is given by

$$T_{\mu\nu} = \frac{i}{4}\{\bar{\Psi}\, \gamma_{(\mu}\nabla_{\nu)}\Psi - \nabla_{(\mu}\bar{\Psi}\,\gamma_{\nu)}\,\Psi\} - \frac{\mu^2}{4}\,\bar{\Psi}\Psi\, g_{\mu\nu} \tag{74}$$

B. The interaction of the fields $A_{\mu\nu}$ and $B_{\mu\nu}$ with the fermions

We will follow a similar procedure in order to describe the coupling of the fermions with the extra two spin-2 fields $A_{\mu\nu}$ and $B_{\mu\nu}$ represented in the Fierz frame. This means that we will make the analysis of the Dirac equation of motion for a spinor embedded in the Cartan geometry.

The covariant derivative is written as above

$$\nabla_\mu\, \Psi \equiv \partial_\mu \Psi - \Gamma_\mu \Psi \tag{75}$$

where now the Fock-Ivanenko spinor connection is given in terms of the affine conection $\Gamma^{\alpha}_{\beta\,\mu}$ given by (34) and (54). The action takes the same form as in equation (72).

Noting that

$$\gamma^5 \equiv -\frac{1}{4!}\eta^{\mu\nu\alpha\lambda}\,\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\lambda, \tag{76}$$

and using the above connection we can re-write the action under the form

$$\begin{aligned} S \;&=\; i\int \sqrt{-g}(\bar{\Psi}\gamma^\alpha\,\nabla_\alpha\,\Psi + \frac{1}{4}\,K_{\mu\nu\alpha}\,\eta^{\mu\nu\alpha\lambda}\,\bar{\Psi}\,\gamma_\alpha\gamma^5\,\Psi)d^4x \\ &-\; \int \sqrt{-g}\mu^2\,\bar{\Psi}\Psi d^4x \end{aligned} \tag{77}$$

From the expression (54) of the contortion we have

$$K_{\mu\nu\alpha}\,\eta^{\mu\nu\alpha\lambda} = \left(2F_{\mu\alpha\nu} + g_{\mu\alpha\nu\lambda}B^\lambda - \eta^{\mu\nu\alpha\lambda}B^\lambda\right)\eta^{\mu\nu\alpha\lambda} \tag{78}$$

which reduces to

$$K_{\mu\nu\alpha}\eta^{\mu\nu\alpha\lambda} = 2B^\lambda \tag{79}$$

Thus the action assumes the form

$$\begin{aligned} S \;&=\; \int \sqrt{-g}(i\bar{\Psi}\gamma^\alpha\,\nabla_\alpha\,\Psi - \mu^2\,\bar{\Psi}\Psi d^4x \\ &+\; \frac{1}{2}\int \sqrt{-g}\,\bar{\Psi}\,\gamma_\lambda\gamma^5\,\Psi\,B^\lambda\,d^4x \end{aligned} \tag{80}$$

which shows that matter minimally coupled to Cartan background geometry interacts only with the $B_{\mu\nu}$ field. Let us analyse a little more carefully such interaction in order to understand the differences of the matter coupling to the spin-2 field which defines the metric tensor in the case of General Relativity.

C. The interaction of spin-2 fields with the fermionic matter

The minimal coupling principle of the matter field Ψ with torsion yields the interacting term (see equation (80))

$$S_{int} = \frac{i}{2}\int \sqrt{-g}\,B^\lambda\,\bar{\Psi}\,\gamma_\lambda\gamma^5\,\Psi d^4x \tag{81}$$

in which we used the decomposition of torsion (51). The quantity B^λ is given in terms of the field $B_{\mu\nu}$ as

$$B_\lambda = \nabla_\mu\,(B^{\alpha\beta}\;g_{\alpha\beta}\delta^\mu_\lambda - B^\mu_\lambda) \tag{82}$$

The interacting term can be re-written, up to a total divergence, as

$$S_{int} = \int \sqrt{-g}\,\{B^{\mu\nu} - B\,g^{\mu\nu}\}\,J_{\mu\nu}d^4x \tag{83}$$

where

$$J_{\mu\nu} = \nabla_\mu(\bar{\Psi}\,\gamma_\nu\gamma^5\,\Psi) \tag{84}$$

The crucial distinction from this coupling (83) to the previous one -(73)concerns the behavior of the current $J_{\mu\nu}$ and $T_{\mu\nu}$ under the operation of hermiticity. The energy-momentum tensor is written as the product of the imaginary quantity i with the difference of two terms containing the derivative operator. Instead of the imaginary quantity, the presence of the hermitian operator γ^5 in the current $J_{\mu\nu}$ is precisely the condition that allows such quantity to be written as a total derivative. This is the crucial point that makes the distinction between these two forms of coupling. In the case of gravity the corresponding term

$$S_{int} = \int \sqrt{-g}\,J_{\mu\nu}g^{\mu\nu}d^4x \tag{85}$$

is a total divergence and thus does not contribute to the dynamics. This is in agreement with the result that gravity is a parity conserving force.

VII. CONCLUSION AND SOME COMMENTS

The purpose of the present paper was to show the existence of two non-equivalent ways to incorporate spin-2 fields into the geometric structure of 4-dimensional spacetime. These modes are related to two distinct representations of spin-2 fields. One of them, the Einstein-frame, uses a symmetric second order tensor $\varphi_{\mu\nu}$ to represent the field; the other, the Fierz-frame, uses a third rank anti-symetric tensor $F_{\alpha\beta\mu}$. In the first case, the natural way to geometrize the field consists in the association of $\varphi_{\mu\nu}$ to the metric tensor, as we showed in equation (1); in

the second case the natural way is the association of the Fierz tensor to the torsion as we did in equation (51). The most important difference between these two representations appears in their formal implementation. In the case of the Einstein-frame one deals with a non-linear structure. This is evident from the expression (2) of the inverse metric. In the Fierz representation instead all the process of geometrization may be linearly implemented. This property led to the choice of the Einstein-frame for the geometrization of the spin-2 field associated to the gravity. All remaining spin-2 fields may be described in the Fierz representation. Besides, in the Fierz frame the traditional ambiguity that was present in coupling higher spin field to gravity is overcomed [4]. Let us point out the property which follows from using the associated Hilbert-Einstein action based on the scalar (67). Although all this structure has a geometrical meaning, if one interprets the extra two spin-2 fields in the realm of Einstein GR as matter fields, then a curious property appears. The equation of motion for each of the extra spin-2 fields is linear and the energy contribution has opposite signs. This allow to recognize the possibility of the existence of a situation in which the energies of both fields cancel out generating a sort of *torsion vacuum state.*

Finally, a remarkable property of such a description appears when we take into account its corresponding interaction with matter. Only one of the fields $B_{\mu\nu}$ has a direct, minimal, coupling with the fermionic field. Besides, such interaction violates parity. This suggests a possible existence of an interaction of this spin-2 field with the matter that violates parity. This is under investigation.

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[8] The proof of this and subsequent lemmas are given in the quoted paper by Novello et al.

[9] there is another one $Z \equiv F_{\alpha\beta\lambda} \, {}^{*F^{\alpha\beta\lambda}}$ which we will not consider here once it is a topological invariant.

[10] Note that we can choose different forms of such restriction of torsion in order to limit it to 20 independent components. In the next section we deal with another choice.

[11] See the appendix for further properties of this description of a spin-2 field in terms of the three-index tensor $F^{\alpha\beta\mu}$.

[12] Note that there is a difference of factor 2 separating our definition of the operator ${G^L}_{\mu\nu}$ and the linear part of the Einstein operator $R_{\mu\nu} - 1/2\, R g_{\mu\nu}$.

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